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Multimode q-Oscillator Algebras with $q^{2(k+1)} = 1$ and Their Bargmann–Fock Representations

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A countably infinite class of multimode q-oscillator algebras $\mathcal{A}_{k;m}$ ($k = 1, 2, \ldots, \infty$; $m = 1, 2, \ldots$) is obtained with the aid of the R-matrix method in quantum group theory for $q^{2(k+1)} = 1$. The related Fock spaces are given and they show that the q-particle systems described by $\mathcal{A}_{k;m}$ obey a generalized Pauli exclusion principle. The algebras $\mathcal{A}_{k;m}$ are represented on a kind of q-holomorphic function spaces $\mathcal{B}(\overline{\eta})_{k;m}$ which are generalizations of the usual Bargmann–Fock spaces with many Grassmann variables and have Hilbert space structures with the scalar product given by an algebraically defined integral. When taking $k = 1$ or $k \to \infty$, all of the above are reduced to the corresponding results for the usual multimode fermion and boson systems, respectively.

1. INTRODUCTION

Owing to their great importance to theoretical and mathematical physics, the q-deformed oscillator algebras have attracted much attention in the last few years (Biedenharn, 1989; Macfarlane, 1989; Hayashi, 1990; Chaichian and Kulish, 1990; Baulieu and Floratos, 1991; Jagannathan *et al.*, 1992; Yang *et al.*, 1997, 1998; and references therein). Among these, there has been increasing interest in the case of deformation parameter q being a root of unity. The single-mode q-oscillator algebra generated by a, $a^\dagger$, and N as

$$aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N}, \qquad [N, a^\dagger] = a^\dagger, \qquad [N, a] = -a \tag{1.1}$$

$$q^s = 1, \qquad s \text{ a positive integer}$$

has been studied (Baulieu and Floratos, 1991; Chaichian and Demichev, 1996; Yang *et al.*, 1997, 1998). Some multimode q-oscillator algebras with

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$q^s = 1$ were also used (Sun and Ge, 1991, 1992; Fu and Ge, 1992) where the operators of different modes are commutative with each other. In this paper, with the aid of the R-matrix method in quantum group theory (Faddeev *et al.*, 1990; Manin, 1988; Wess and Zumino, 1990; Brzezinski *et al.*, 1992), we set up a new kind of multimode q-oscillator algebra in which the operators of different modes are q-commutative rather than commutative, so that they contain the usual fermionic and bosonic multimode oscillator algebras as special cases. Moreover, we also construct Fock spaces and Bargmann–Fock representations of these multimode q-oscillator algebras and discuss some related problems.

In Section 2, we give a consistent differential calculus on m-dimensional q-spaces with $q^{2(k+1)} = 1$, $k = 1, 2, \ldots$. These are used in Section 3 to construct a countably infinite set of multimode q-oscillator algebras. Their Fock spaces are also obtained and it is found that the q-particle systems described by these q-oscillator algebras obey a generalized Pauli exclusion principle. Section 4 is devoted to their Bargmann–Fock representations on some q-holomorphic function spaces with many generalized Grassmann variables; we show that these spaces are Hilbert ones through some scalar products given by algebraically defined integrals. When taking $k = 1$ or $k \to \infty$, all of the above become the results corresponding to the usual fermionic or bosonic multimode cases, respectively. Some conclusions and discussions are given in Section 5.

2. *q*-DIFFERENTIAL CALCULUS ON *q*-SPACES WITH $q^{2(k+1)} = 1$

To obtain a multimode generalization of the q-algebra (1.1), we start with some differential calculus on the m-dimensional q-space. According to Wess and Zumino (1990), given matrices $B, C, F \in \mathrm{End}(\mathbf{C}^m \otimes \mathbf{C}^m)$ fulfilling the consistency conditions

$$\begin{aligned} &B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23}, \qquad C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23} \\ &B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23}, \qquad (B_{12} - I)(C_{12} + I) = 0 \end{aligned} \tag{2.1a}$$

$$\begin{aligned} &F_{12}F_{23}F_{12} = F_{23}F_{12}F_{23}, \\ &C_{12}C_{23}F_{12} = F_{23}C_{12}C_{23}, \qquad (C_{12} + I)(F_{12} - I) = 0 \end{aligned} \tag{2.1b}$$

with $B_{12} = B \otimes I$, $B_{23} = I \otimes B$, etc., we can construct a consistent differential calculus on a q-algebra (called a q-space) generated by $\{x^i, i = 1, \ldots, m\}$ such that

$$\begin{aligned} &x_1x_2 = B_{12}x_1x_2, \qquad x_1dx_2 = C_{12}(dx_1)x_2, \qquad dx_1\, dx_2 = -C_{12}\, dx_1\, dx_2 \\ &\mathrm{D}_2\mathrm{D}_1 = \mathrm{D}_2\mathrm{D}_1F_{12}, \qquad \mathrm{D}_jx^i = \delta^i_j + C^{ik}_{jl}x^l\,\mathrm{D}_k, \qquad \mathrm{D}_j\, dx^i = (C^{-1})^{ik}_{jl}\, dx^l\,\mathrm{D}_k \end{aligned} \tag{2.2}$$

where standard tensor notations are used and dx^i and D_i, $i = 1, \ldots, m$, are referred to as q-(exterior) differentials and q-derivatives, respectively.

The above x^i, dx^i, D_i, $i = 1, \ldots, m$, are generally regarded as algebraic elements; for our present purpose, we need only to consider the subalgebra generated by $\{x^i, \mathrm{D}_i, i = 1, \ldots, m\}$. Now we look for a set of matrices B, C, F satisfying (2.1) and suitable for our construction of multimode q-algebras. Motivated by Brzezinski *et al.* (1992), we take

$$\begin{aligned} B = F &= \sum_i e^i_i \otimes e^i_i + q^2 \sum_{i<j} e^i_j \otimes e^j_i + q^{-2} \sum_{i>j} e^i_j \otimes e^j_i \\ C(p) &= \sum_i p_i\, e^i_i \otimes e^i_i + q^2 \sum_{i<j} e^i_j \otimes e^j_i + q^{-2} \sum_{i>j} e^i_j \otimes e^j_i \end{aligned} \tag{2.3}$$

where $e^i_j \in \mathrm{Mat}\,(\mathbf{C}^m)$ are matrix units, the symbol p in $C(p)$ stands for multiple parameter $\{p_i \in \mathbf{C}, i = 1, \ldots, m\}$, and q is a root of unity such that

$$q^{2(k+1)} = 1 \quad (\text{i.e., } q = e^{\pi i/(k+1)}), \qquad k = 1, 2, \ldots, \infty \tag{2.4}$$

The fact that B, $C(p)$ satisfy (2.1a) has been pointed out by Brzezinski *et al.* (1992); here we show that, for our selection of q, the conditions (2.1b) are also satisfied.

Since $F = B$, we need only to verify the second and third equations in (2.1b). From (2.3) and (2.4), it follows that

$$F^\dagger\ (=B^\dagger) = F\ (=B), \qquad C^\dagger(p) = C(p^*) \tag{2.5}$$

where the dagger stands for Hermitian conjugation of the matrices. Because p_i, $i = 1, \ldots, m$, and q in (2.3) are independent of each other, by (2.1a) and (2.5) we have

$$\begin{aligned} & B_{12}C_{23}(p)C_{12}(p) = C_{23}(p)C_{12}(p)B_{23} \\ \Leftrightarrow\ & B_{12}C_{23}(p^*)C_{12}(p^*) = C_{23}(p^*)C_{12}(p^*)B_{23} \\ \underset{B=F}{\overset{\dagger}{\Leftrightarrow}}\ & C_{12}(p)C_{23}(p)F_{12} = F_{23}C_{12}(p)C_{23}(p) \end{aligned} \tag{2.6a}$$

and

$$\begin{aligned} & (B_{12} - I)(C_{12}(p) + I) = 0 \\ \Leftrightarrow\ & (B_{12} - I)(C_{12}(p^*) + I) = 0 \\ \underset{B=F}{\overset{\dagger}{\Leftrightarrow}}\ & (C_{12}(p) + I)(F_{12} - I) = 0 \end{aligned} \tag{2.6b}$$

Thus by (2.3) and (2.2) we can write explicitly the algebraic relations of x^i, D_i, $i = 1, \ldots, m$, as follows:

$$
\begin{gathered}
x^i x^j = q^2 x^j x^i, \qquad \mathrm{D}_i \mathrm{D}_j = q^2 \mathrm{D}_j \mathrm{D}_i \\
\mathrm{D}_j x^i = q^2 x^i \mathrm{D}_j, \qquad \mathrm{D}_i x^j = q^{-2} x^j \mathrm{D}_i \qquad i < j \\
\mathrm{D}_i x^i - p_i x^i \mathrm{D}_i = 1 \qquad \text{(no sum on the index } i\text{)} \\
i, j = 1, \ldots, m
\end{gathered} \tag{2.7}
$$

So far, the parameters p_i are arbitrary. Now we take $p_i = q^2$, $i = 1, \ldots, m$; then, (2.7) gives

$$
\begin{gathered}
x^i x^j = q^2 x^j x^i, \qquad \mathrm{D}_i \mathrm{D}_j = q^2 \mathrm{D}_j \mathrm{D}_i \\
\mathrm{D}_j x^i = q^2 x^i \mathrm{D}_j, \qquad \mathrm{D}_i x^j = q^{-2} x^j \mathrm{D}_i \qquad i < j \\
\mathrm{D}_i x^i - q^2 x^i \mathrm{D}_i = 1 \qquad \text{(no sum on the index } i\text{)} \\
i, j = 1, \ldots, m
\end{gathered} \tag{2.8}
$$

The positive integer k in (2.4) is closely related to the features of the algebras discussed in this paper, so we shall call such algebras q_k-algebras.

3. MULTIMODE q_k-OSCILLATOR ALGEBRAS AND THEIR FOCK SPACES

Let us introduce the raising and lowering operators $\{\bar{b}_i, b_i\}$ by the correspondence

$$
x^i \to \bar{b}_i, \qquad \mathrm{D}_i \to b_i, \qquad i = 1, \ldots, m \tag{3.1}
$$

Then from (2.8) their commutation relations

$$
\begin{gathered}
\bar{b}_i \bar{b}_j = q^2 \bar{b}_j \bar{b}_i, \qquad b_i b_j = q^2 b_j b_i \\
b_j \bar{b}_i = q^2 \bar{b}_i b_j, \qquad b_i \bar{b}_j = q^{-2} \bar{b}_j b_i \qquad i < j \\
b_i \bar{b}_i - q^2 \bar{b}_i b_i = 1 \qquad \text{(no sum on } i\text{)} \\
i, j = 1, \ldots, m
\end{gathered} \tag{3.2}
$$

follow immediately. The discussions of Section 2 show that (3.2) gives an algebra over field **C**. Moreover, from (3.2) we have

$$
\begin{aligned}
&\bar{b}_i^{k+1} \bar{b}_j = q^{2(k+1)} \bar{b}_j \bar{b}_i^{k+1}, && b_j \bar{b}_i^{k+1} = q^{2(k+1)} \bar{b}_i^{k+1} b_j, \qquad i < j \\
&\bar{b}_i^{k+1} \bar{b}_l = q^{-2(k+1)} \bar{b}_l \bar{b}_i^{k+1}, && b_l \bar{b}_i^{k+1} = q^{-2(k+1)} \bar{b}_i^{k+1} b_l, \qquad i > l \\
&b_i \bar{b}_i^{k+1} = q^{2(k+1)} \bar{b}_i^{k+1} b_i + [k+1]_{q^2} \bar{b}_i^k, && i, j = 1, \ldots, m
\end{aligned} \tag{3.3}
$$

where $[k + 1]_{q^2} \equiv 1 + q^2 + \cdots + q^{2k} = (q^{2(k+1)} - 1)/(q^2 - 1)$. Thus (2.4) and (3.3) imply that $[k + 1]_{q^2} = 0$ and $\bar{b}_i^{k+1}$, $i = 1, \ldots, m$, are central elements of the algebra (3.2). Similarly, b_i^{k+1}, $i = 1, \ldots, m$, are also central elements. As in the single-mode case (Baulieu and Floratos, 1991), we take consistently

$$\bar{b}_i^{k+1} = 0, \qquad b_i^{k+1} = 0, \qquad i = 1, \ldots, m \tag{3.4}$$

When $k = 1$ and $k \to \infty$, the relations (3.2) and (3.4) give, respectively, the usual fermionic and bosonic multimode algebras.

For physical applications, it is convenient to introduce the number operators N_i, $i = 1, \ldots, m$, such that

$$\begin{gathered}[N_i, \bar{b}_j] = \delta_{ij}\bar{b}_j, \qquad [N_i, b_j] = -\delta_{ij} b_j, \\ [N_i, N_j] = 0, \qquad i, j = 1, \ldots, m\end{gathered} \tag{3.5}$$

From (3.2) and (3.4), it can be verified that N_i can be expressed in terms of $\bar{b}_i$, b_i as

$$N_i = \sum_{l=1}^{k} \frac{(1 - q^2)^l}{(1 - q^{2l})} \bar{b}_i^l b_i^l, \qquad i = 1, \ldots, m \tag{3.6}$$

This shows that N_i are also elements of the algebra defined by (3.2) and (3.4). Even so, for convenience of discussion, we shall still regard, equivalently, the algebra as being generated by $\{\bar{b}_i, b_i, N_i, i = 1, \ldots, m\}$ with relations (3.2), (3.4), and (3.5).

It should be noted that $\bar{b}_i$ and b_i are not Hermitian conjugate to each other except for the cases $k = 1$ $(q = i)$ and $k \to \infty$ $(q = 1)$. As for N_i, at least to the Fock-type representations, they are Hermitian (for the single-mode case, see Baulieu and Floratos, 1991; Oh and Singh, 1994; Quesne and Vansteenkiste, 1995). To circumvent the problem of $b_i^\dagger \neq \bar{b}_i$, $(\bar{b}_i)^\dagger \neq b_i$, we introduce the a-creation and annihilation operators a_i, $a_i^\dagger$ as

$$a_i = q^{-N_i/2} b_i, \qquad a_i^\dagger = \bar{b}_i q^{-N_i/2}, \qquad i = 1, \ldots, m \tag{3.7}$$

and from (3.2), (3.4), and (3.5) we obtain

$$\begin{gathered}a_i a_j = q^2 a_j a_i, \qquad a_i^\dagger a_j^\dagger = q^2 a_j^\dagger a_i^\dagger, \\ a_j a_i^\dagger = q^2 a_i^\dagger a_j, \qquad i < j \\ [N_i, a_j] = -\delta_{i,j} a_j, \qquad [N_i, a_j^\dagger] = \delta_{i,j} a_j^\dagger, \\ [N_i, N_j] = 0, \qquad i, j = 1, \ldots, m \\ a_i a_i^\dagger - q^{\pm 1} a_i^\dagger a_i = q^{\mp N_i}, \qquad (a_i^\dagger)^{k+1} = 0, \\ a_i^{k+1} = 0\end{gathered} \tag{3.8}$$

Here a_i, $a_i^\dagger$ are Hermitian conjugates of each other and as useful consequences we also have

$$a_i^\dagger a_i = [N_i], \qquad a_i a_i^\dagger = [N_i + 1], \qquad i = 1, \ldots, m \tag{3.9}$$

where we have used the abbreviation

$$[X] := \frac{q^X - q^{-X}}{q - q^{-1}} \tag{3.10}$$

for an operator or a number X. (For the single-mode case, see Biedenharn, 1989; Macfarlane, 1989; Baulieu and Floratos, 1991). Relations (3.8) give a countably infinite set of multimode q_k-oscillator algebras, which will be denoted by $\mathcal{A}_{k;m}$ ($k = 1, 2, \ldots, \infty$; $m = 1, 2, \ldots$). It is these $\mathcal{A}_{k;m}$ that we want to obtain and discuss in this paper.

The Fock representation of $\mathcal{A}_{k;m}$ can be constructed as follows. Let $|0\rangle = |0, \ldots, 0\rangle$ be the vacuum state defined by

$$N_i|0\rangle = 0, \qquad a_i|0\rangle = 0, \qquad i = 1, \ldots, m \tag{3.11}$$

and let $\{|n_1, \ldots, n_m\rangle, n_1, \ldots, n_m = 0, 1, \ldots\}$ be the orthonormal set of common eigenstates of $\{N_i, i = 1, \ldots, m\}$ such that

$$N_i|n_1, \ldots, n_i, \ldots, n_m\rangle = n_i|n_1, \ldots, n_i, \ldots, n_m\rangle \tag{3.12}$$

$$\langle n_1', \ldots, n_i', \ldots, n_m'|n_1, \ldots, n_i, \ldots, n_m\rangle = \prod_{i=1}^{m} \delta_{n_i' n_i}$$

Thus from (3.8) we can obtain

$$a_j|n_1, \ldots, n_j, \ldots, n_m\rangle = q^{2\Sigma_{i<j} n_i}[n_j]^{1/2}|n_1, \ldots, n_j - 1, \ldots, n_m\rangle \tag{3.13}$$

$$a_j^\dagger|n_1, \ldots, n_j, \ldots, n_m\rangle = q^{-2\Sigma_{i<j} n_i}[n_j + 1]^{1/2}|n_1, \ldots, n_j + 1, \ldots, n_m\rangle$$

Since $[k + 1] = 0$ by (3.10) and (2.4), then (3.13) show that if some $n_j = k$, we have $a_j^\dagger|n_1, \ldots, n_j = k, \ldots, n_m\rangle = 0$. Therefore, $\{|n_1, \ldots, n_m\rangle, n_1, \ldots, n_m = 0, 1, \ldots, k\}$ constitute a basis of the Fock space and the equations (3.12), (3.13) give a finite-dimensional q_k-Fock representations of $\mathcal{A}_{k;m}$. Moreover, from (3.8) and (3.11)–(3.13) we can write

$$|n_1, \ldots, n_i, \ldots, n_m\rangle = \frac{(a_1^\dagger)^{n_1} \cdots (a_i^\dagger)^{n_i} \cdots (a_m^\dagger)^{n_m}}{([n_1]! \cdots [n_i]! \cdots [n_m]!)^{1/2}}|0\rangle \tag{3.14}$$

$$n_1, \ldots, n_m = 0, 1, \ldots, k.$$

Noting the fact $[n] > 0$ for $1 \le n \le k$ by (3.10), we see that the Fock space given by (3.11)–(3.13) is positive definite.

Physically, for each fixed, positive integer value of k, $\mathcal{A}_{k;m}$ can be considered as describing a kind of particle which we will call q_k-particles, and in from the viewpoint of the occupation number, (3.11)–(3.14) show that in the q_k-particle system, a quantum state may hold at most k identical q_k-particles. This is a generalization of the usual Pauli exclusion principle. When $k = 1$ or $k \to \infty$, these give the well-known results of the fermionic or bosonic multimode systems, repectively. When $2 \leq k < \infty$, we obtain a new kind of multimode q_k-oscillator algebra which describe, the so-called q_k-particle systems obeying statistics different from Bose and Fermi ones. The statistical properties of these new q_k-particle systems will be discussed elsewhere.

4. BARGMANN–FOCK REPRESENTATION AND HILBERT SPACE STRUCTURE OF THE q_k-HOLOMORPHIC FUNCTION SPACE

Let $\overline{\eta}_1, \ldots, \overline{\eta}_m$ be an ordered set of q_k-commutative coordinates. We intend to obtain a q_k-Bargmann–Fock represenation ρ of $\mathscr{A}_{k;m}$ on the so-called q_k-holomorphic function space generated by $\{\overline{\eta}_i, i = 1, \ldots, m\}$:

$$\rho: \quad a^\dagger \to \overline{\eta}_i, \quad a_i \to \partial_{\overline{\eta}_i} \equiv \overline{\partial}_i, \quad N_i \to \overline{\mathcal{N}}_i \tag{4.1}$$

Thus from (3.8) we immediately get

$$\begin{gathered}
\overline{\eta}_i\overline{\eta}_j = q^2\,\overline{\eta}_j\overline{\eta}_i, \qquad \overline{\eta}_i^{k+1} = 0 \\
\overline{\partial}_i\overline{\partial}_j = q^2\overline{\partial}_j\overline{\partial}_i, \qquad \overline{\partial}_i^{k+1} = 0, \qquad i < j \\
\overline{\partial}_i\overline{\eta}_i - q^{\pm1}\,\overline{\eta}_i\overline{\partial}_i = q^{\mp\overline{\mathcal{N}}_i}, \qquad \overline{\partial}_i\overline{\eta}_j = q^{-2}\,\overline{\eta}_j\overline{\partial}_i, \qquad \overline{\partial}_j\,\overline{\eta}_i = q^2\,\overline{\eta}_i\overline{\partial}_j \\
[\overline{\mathcal{N}}_i, \overline{\eta}_j] = \delta_{ij}\overline{\eta}_j, \qquad [\overline{\mathcal{N}}_i, \overline{\partial}_j] = -\delta_{ij}\overline{\partial}_j, \qquad [\overline{\mathcal{N}}_i, \overline{\mathcal{N}}_j] = 0 \\
i, j = 1, \ldots, m
\end{gathered} \tag{4.2}$$

The first line of (4.2) also gives the commutation relations of the generators $\overline{\eta}_1, \ldots, \overline{\eta}_m$ of the q_k-holomorphic function space; this space is a generalization of the usual Bargmann–Fock space with many Grassmann variables and will be denoted by $\mathscr{B}(\overline{\eta})_{k;m}$ ($k = 1, 2, \ldots, \infty$; $m = 1, 2, \ldots$). For each pair of fixed (k, m), the finite set $\{\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}, n_1, \ldots, n_m = 0, 1, \ldots, k\}$ constitutes a linear basis of $\mathscr{B}(\overline{\eta})_{k;m}$, and a q_k-holomorphic function $f(\overline{\eta}) \in \mathscr{B}(\overline{\eta})_{k;m}$ can be expanded as

$$f(\overline{\eta}) = \sum_{n_1,\ldots,n_m=0}^{k} f_{n_i,\ldots,n_m}\,\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m} \tag{4.3}$$

where $f_{n_1,\ldots,n_m} \in \mathbf{C}$. This will be called the standard form of $f(\overline{\eta})$.

In order to give the explicit form of the q_k-Bargmann–Fock representation of $\mathscr{A}_{k;m}$ on $\mathscr{B}(\overline{\eta})_{k;m}$ conveniently, we introduce another set of derivaties

$\{\hat{\partial}_{\overline{\eta}_i} \equiv \hat{\bar{\partial}}_i,\ i = 1, \ldots, m\}$ which correspond to equation (2.7) with $p_i = 1$, $i = 1, \ldots, m$, and obey the relations

$$\begin{aligned} &\hat{\bar{\partial}}_i\, \hat{\bar{\partial}}_j = q^2 \hat{\bar{\partial}}_j\, \hat{\bar{\partial}}_i, \qquad \hat{\bar{\partial}}_i \overline{\eta}_j = q^{-2} \overline{\eta}_j \hat{\bar{\partial}}_i, \qquad \hat{\bar{\partial}}_j \overline{\eta}_i = q^2 \overline{\eta}_i \hat{\bar{\partial}}_j, \qquad i < j \\ &\hat{\bar{\partial}}_i \overline{\eta}_i - \overline{\eta}_i \hat{\bar{\partial}}_i = 1 \\ &i, j = 1, \ldots, m. \end{aligned} \tag{4.4}$$

Thus, when $\hat{\bar{\partial}}$ is acting on $f(\overline{\eta}) \in \mathcal{B}(\overline{\eta})_{k;m}$, we have

$$\hat{\bar{\partial}}_i f(\overline{\eta}) = \lim_{\epsilon \to 0} \frac{1}{\overline{\eta}_i} \frac{f(\overline{\eta}_1, \ldots, (1+\epsilon)\overline{\eta}_i, \ldots, \overline{\eta}_m) - f(\overline{\eta}_1, \ldots, \overline{\eta}_i, \ldots, \overline{\eta}_m)}{\epsilon}$$
$$i = 1, \ldots, m \tag{4.5}$$

where $\epsilon \neq 0$ is real.

With (4.4) and (4.5), the algebra $\mathcal{A}_{k;m}$ can be represented on $\mathcal{B}(\overline{\eta})_{k;m}$ as

$$\begin{aligned} \rho(a_i^\dagger) f(\overline{\eta}) &= \overline{\eta}_i f(\overline{\eta}_1, \ldots, \overline{\eta}_m) \\ \rho(a_i) f(\overline{\eta}) &= \bar{\partial}_i f(\overline{\eta}_1, \ldots, \overline{\eta}_m) \\ &\equiv \frac{1}{\overline{\eta}_i} \frac{q^{\overline{\eta}_i \hat{\bar{\partial}}_i} - q^{-\overline{\eta}_i \hat{\bar{\partial}}_i}}{q - q^{-1}} f(\overline{\eta}_1, \ldots, \overline{\eta}_i, \ldots, \overline{\eta}_m) \\ &= \frac{1}{\overline{\eta}_i} \frac{f(\overline{\eta}_1, \ldots, q\overline{\eta}_i, \ldots, \overline{\eta}_m) - f(\overline{\eta}_1, \ldots, q^{-1}\overline{\eta}_i, \ldots, \overline{\eta}_m)}{q - q^{-1}} \\ \rho(N_i) f(\overline{\eta}) &= \overline{\mathcal{N}}_i f(\overline{\eta}_1, \ldots, \overline{\eta}_m) \\ &= \overline{\eta}_i \hat{\bar{\partial}}_i f(\overline{\eta}_1, \ldots, \overline{\eta}_i, \ldots, \overline{\eta}_m) \\ i &= 1, \ldots, m \end{aligned} \tag{4.6}$$

More concretely, to the element $\overline{\eta}_1^{n_1} \ldots \overline{\eta}_m^{n_m}$ in the basis of $\mathcal{B}(\overline{\eta})_{k;m}$, we have

$$\begin{aligned} \overline{\eta}_i(\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}) &= q^{-2\Sigma_{l<i} n_l} \overline{\eta}_1^{n_1} \cdots \overline{\eta}_i^{n_i+1} \cdots \overline{\eta}_m^{n_m} \\ \bar{\partial}_i(\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}) &= q^{2\Sigma_{l<i} n_l} [n_i] \overline{\eta}_1^{n_1} \cdots \overline{\eta}_i^{n_i-1} \cdots \overline{\eta}_m^{n_m} \\ \overline{\mathcal{N}}_i(\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}) &= n_i \overline{\eta}_1^{n_1} \cdots \overline{\eta}_i^{n_i} \cdots \overline{\eta}_m^{n_m} \\ 0 \le n_i \le k, &\qquad i = 1, \ldots, m \end{aligned} \tag{4.7}$$

and these can be extended to the whole $\mathcal{B}(\overline{\eta})_{k;m}$ by linearity.

Noticing that $q^* = q^{-1}$, we introduce consistently the complex map, denoted by a bar, as

$$^{-}: \quad \overline{\eta}_i \to \eta_i, \qquad \bar{\partial}_i \to \partial_i \equiv \partial_{\eta_i}, \qquad \overline{\mathcal{N}}_i \to \mathcal{N}_i, \qquad \overline{AB} = \overline{A}\overline{B}, \ ^{-2} = 1 \tag{4.8}$$

where $A, B \in \mathcal{B}(\overline{\eta})_{k;m}$ or $\mathcal{A}_{k;m}$ and for the coefficients of the polynomials, the bar operation means ordinary complex conjugation. The complex conjugation of (4.2), (4.4) gives

$$\begin{aligned}
&\eta_i\overline{\eta}_j = q^{-2}\eta_j\eta_i, \qquad \eta_i^{k+1} = 0\\
&\partial_i\partial_j = q^{-2}\partial_j\partial_i, \qquad \partial_i^{k+1} = 0, \qquad i < j\\
&\partial_i\eta_i - q^{\mp 1}\eta_i\partial_i = q^{\mp\mathcal{N}_i}, \qquad \partial_i\eta_j = q^2\eta_j\partial_i, \qquad \partial_j\eta_i = q^{-2}\eta_i\partial_j \qquad (4.9)\\
&[\mathcal{N}_i, \eta_j] = \delta_{ij}\eta_j, \qquad [\mathcal{N}_i, \partial_j] = -\delta_{ij}\partial_j, \qquad [\mathcal{N}_i, \mathcal{N}_j] = 0\\
&i, j = 1, \ldots, m
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\partial}_i\hat{\partial}_j = q^{-2}\hat{\partial}_j\hat{\partial}_i, \qquad \hat{\partial}_i\eta_j = q^2\eta_j\hat{\partial}_i, \qquad \hat{\partial}_j\eta_i = q^{-2}\eta_i\hat{\partial}_j, \qquad i < j\\
&\hat{\partial}_i\eta_i - \eta_i\hat{\partial}_i = 1 \qquad (4.10)\\
&i, j = 1, \ldots, m
\end{aligned}$$

Then for

$$h(\eta) = \sum_{n_1,\ldots,n_m=0}^{k} h_{n_1,\ldots,n_m}\eta_1^{n_1}\cdots\eta_m^{n_m}$$

we have

$$\partial_i h(\eta) = \frac{1}{\eta_i}\frac{h(\eta_1, \ldots, q\eta_i, \ldots, \eta_m) - h(\eta_1, \ldots, q^{-1}\eta_i, \ldots, n_m)}{q - q^{-1}} \qquad (4.11)$$

$$\mathcal{N}_i h(\eta) = \eta_i\hat{\partial}_i h(\eta) = \sum_{n_1,\ldots,n_m=0}^{k} n_i h_{n_1,\ldots,n_m}\eta_1^{n_1}\cdots\eta^{n_m}$$

For the mixed commutation relations between the barred and unbarred entities we take

$$\begin{aligned}
&\overline{\eta}_i\eta_j = q^2\eta_j\overline{\eta}_i, \qquad \overline{\partial}_i\partial_j = q^2\partial_j\overline{\partial}_i \qquad (4.12)\\
&\overline{\partial}_i\eta_j = q^{-2}\eta_j\overline{\partial}_i, \qquad [\overline{\mathcal{N}}_i, \mathcal{N}_j] = 0; \qquad i, j = 1, \ldots, m
\end{aligned}$$

If we introduce the extended symbols $\bar{z}_l$, $\overline{\nabla}_l$, $\overline{\aleph}$, $l, = 1, 2, \ldots, 2m$, such that

$$\begin{aligned}
&\bar{z}_i = \overline{\eta}_i, \qquad \bar{z}_{m+i} = \eta_{m+1-i}\\
&\overline{\nabla}_i = \overline{\partial}_i, \qquad \overline{\nabla}_{m+i} = \partial_{m+1-i}, \qquad i = 1, \ldots, m \qquad (4.13)\\
&\overline{\aleph}_i = \overline{\mathcal{N}}_i, \qquad \overline{\aleph}_{m+i} = \mathcal{N}_{m+1-i}
\end{aligned}$$

it follows that

$$^{-}: \quad \bar{z}_i \to z_l = \bar{z}_{2m+1-l}, \qquad \overline{\nabla}_l \to \nabla_l = \overline{\nabla}_{2m+1-l}, \qquad \overline{\aleph}_l \to \aleph_l = \overline{\aleph}_{2m+1-l}$$
$$l = 1, 2, \ldots, 2m \tag{4.14}$$

and the commutation relations (4.2), (4.9), and (4.12) can be expressed in a unified form

$$\begin{aligned} &\bar{z}_l\bar{z}_s = q^2\bar{z}_s\bar{z}_l, \qquad \bar{z}_l^{k+1} = 0 \\ &\overline{\nabla}_l\overline{\nabla}_s = q^2\overline{\nabla}_s\overline{\nabla}_l, \qquad \overline{\nabla}_l^{k+1} = 0, \qquad l < s \\ &\overline{\nabla}_l\bar{z}_l - q^{\pm 1}\bar{z}_l\overline{\nabla}_l = q^{\mp\overline{\aleph}_l}, \qquad \overline{\nabla}_l\bar{z}_s = q^{-2}\bar{z}_s\overline{\nabla}_t, \qquad \overline{\nabla}_s\bar{z}_l = q^2\bar{z}_l\overline{\nabla}_s \\ &[\overline{\aleph}_i, \bar{z}_s] = \delta_{ls}\bar{z}_s, \qquad [\overline{\aleph}_l, \overline{\nabla}_s] = -\delta_{ls}\overline{\nabla}_s, \qquad [\overline{\aleph}_l, \overline{\aleph}_s] = 0 \\ &l, s = 1, \ldots, 2m \end{aligned} \tag{4.15}$$

which are essentially the relations (4.2) with the extension $m \to 2m$. Furthermore, the algebraic relations (4.15) [or equivalently (4.2), (4.9), (4.12) as a whole] are invariant under the complex conjugation $^{-}$. Thus, from the discussion in Sections 2 and 3, the algebraic relations (4.2), (4.9), and (4.12) are compatible with each other and the bar operation is a well-defined involutive map on them.

To construct a scalar product on $\mathcal{B}(\overline{\eta})_{k;m}$, we define the *evaluation* of differentiation in a way consistent with all algebraic structures and the complex conjugation $^{-}$: All $\bar{\partial}_i$, ∂_i are to be commuted to the right (through the algebraic relations). When they arrive, the corresponding terms are to be set equal to zero owing to $\bar{\partial}_i(1) = \partial_i(1) = 0$. What remains is the value of the differentiation.

Now we define the scalar product $\langle\cdot,\cdot\rangle$ of $f(\overline{\eta})$, $g(\overline{\eta}) \in \mathcal{B}(\overline{\eta})_{k;m}$ by the following "integral":

$$\langle f, g\rangle := [e_q^{\bar{\partial}_m\partial_m}e^{\bar{\partial}_{m-1}}\partial_q^{m-1} \ldots e_q^{\bar{\partial}_1\partial_1}(\overline{f(\overline{\eta})})^{\top}g(\overline{\eta})]_{\text{evaluated at } \overline{\eta}=0=\eta} \tag{4.16}$$

where

$$e_q^{\bar{\partial}_i\partial_i} \equiv \sum_{n=0}^{k}\frac{(\bar{\partial}_i\partial_i)^n}{[n]_{q^2}!} = \sum_{n=0}^{k}\frac{\bar{\partial}_i^n\partial_i^n}{[n]!}, \qquad [n]_{q^2} \equiv \frac{q^{2n}-1}{q^2-1}$$

and $(\overline{f(\eta)})^{\top}$ stands for the "bar"-Hermitian conjugation of $f(\overline{\eta})$, which is defined as follows: first write $f(\overline{\eta})$ in the standard form [cf. (4.3)] and then take the Hermitian conjugation, i.e., for $f(\overline{\eta})$ as in (4.3), we have

$$(\overline{f(\overline{\eta})})^{\top} = \sum_{n_1,\ldots,n_m=0}^{k} f^*_{n_1,\ldots,n_m}\eta_m^{n_m}\ldots\eta_1^{n_1} \tag{4.17}$$

Thus from (4.2), (4.9), (4.12), (4.16), and (4.17), it follows that

$$\langle f, g\rangle = \sum_{n_1,\ldots,n_m=0}^{k} f^*_{n_1,\ldots,n_m} g_{n_1,\ldots,n_m}[n_1]![n_2]! \ldots [n_m]! \tag{4.18}$$

where we have written

$$g(\overline{\eta}) = \sum_{n_1,\ldots,n_m=0}^{k} g_{n_1,\ldots,n_m} \overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}, \qquad g_{n_1,\ldots,n_m} \in \mathbf{C}$$

Since for $1 \leq n \leq k$, $[n] > 0$, the scalar product (4.16) is positive definite. The q_k-holomorphic function space $\mathcal{B}(\overline{\eta})_{k;m}$ with this inner product is a Hilbert space and the set

$$\left\{ \frac{\overline{\eta}_1^{n_1} \cdots \overline{\eta}_m^{n_m}}{([n_1]! \cdots [n_m]!)^{1/2}}, n_1, \ldots, n_m = 0, 1, \ldots, k \right\}$$

is its orthonormal basis. When restricted within the single-mode case, (4.16) gives the inner product similar to that of Baulieu and Floratos (1991), but by a different integral expression. The algebraically defined integral here has the advantages that it avoids the discussions of q-(exterior) differentials and that its evaluation procedure is the same in all cases of $k = 1, 2, \ldots, \infty$. Moreover, (the representations of) a_i and $a_i^\dagger$ on $\mathcal{B}(\overline{\eta})_{k;m}$ are adjoint with each other in respect to (4.16). Indeed, by some straightforward but tedious calculations we have

$$\langle \overline{\partial}_i f, g\rangle = \langle f, \overline{\eta}_i g\rangle, \qquad \langle \overline{\eta}_i f, g\rangle = \langle f, \overline{\partial}_i g\rangle \tag{4.19}$$

for all $f(\overline{\eta}), g(\overline{\eta}) \in \mathcal{B}(\overline{\eta})_{k;m}$.

5. CONCLUSION AND DISCUSSION

A countably infinite class of the multimode q_k-oscillator algebras $\mathcal{A}_{k;m}$ ($k = 1, 2, \ldots, \infty$; $m = 1, 2, \ldots$) was obtained with the aid of the R-matrix method in quantum group theory. The algebras $\mathcal{A}_{k;m}$ are a kind of generalization of the usual multimode fermionic and bosonic algebras. The Fock spaces of the $\mathcal{A}_{k;m}$ were given and showed that the q_k-particle systems described by $\mathcal{A}_{k;m}$ obey some generalized Pauli exclusion principle. As one of the main results, we also constructed the q_k-Bargmann–Fock representations of the $\mathcal{A}_{k;m}$ on a kind of q_k-holomorphic function spaces $\mathcal{B}(\overline{\eta})_{k;m}$ which are generalizations of Grassmann algebras with many variables. The scalar product on $\mathcal{B}(\overline{\eta})_{k;m}$ was given by an algebraically defined integral which is positive definite and endows the $\mathcal{B}(\overline{\eta})_{k;m}$ with Hilbert space structures. When taking $k = 1$ or $k = \infty$, all of the above reduce to the corresponding results in the usual quantum field theories (Itzykson and Zuber, 1980) for the fermion and boson systems, respectively. In some sense, the results of this paper may also be regarded as the first step for constructing a q_k-quantum field theory describing the q_k-particle systems. Further construction of this field theory needs additional consideration.

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REFERENCES

L. Baulieu and E. G. Floratos (1991). *Physics Letters B*, **258**, 171.
L. C. Biedenharn (1989). *Journal of Physics A: Mathematical and General*, **22**, L873.
T. Brzezinski, H. Danrowski, and J. Rembielinski (1992). *Journal of Mathematical Physics*, **33**, 19.
M. Chaichian and A. Demichev (1996). *Introduction to Quantum Groups*, World Scientific, Singapore.
M. Chaichian and P. P. Kulish (1990). *Physics Letters B*, **234**, 72.
L. Faddeev, L. Takhtajan, and N. Reshetikhin (1989). *Leningrad Mathematical Journal*, **1**, 193.
H. C. Fu and M. L. Ge (1992). *Journal of Mathematical Physics*, **33**, 427.
T. Hayashi (1990). *Communications in Mathematical Physics*, **127**, 129.
C. Itzykson and J-B. Zuber (1980). *Quantum Field Theory*, McGraw-Hill, New York.
R. Jagannathan, *et al.* (1992). *Journal of Physics A: Mathematical and General*, **25**, 6429.
A. J. Macfarlane (1989). *Journal of Physics A: Mathematical and General*, **22**, 4581.
Yu. I. Manin (1988). Quantum groups and non-commutative geometry. Technical Report, Centre de Recherches Mathematiques, Montreal.
C. H. Oh and K. Singh (1994). *Journal of Physics A: Mathematical and General*, **27**, 5907.
C. Quesne and N. Vansteenkiste (1995). *Journal of Physics A: Mathematical and General*, **28**, 7019.
C. P. Sun and M. L. Ge (1991). *Journal of Physics A: Mathematical and General*, **24**, 3265.
C. P. Sun and M. L. Ge (1992). *Journal of Physics A: Mathematical and General*, **25**, 401.
J. Wess and B. Zumino (1990). *Nuclear Physics B (Proceedings Supplement)*, **18**, 302.
Y. P. Yang, W. G. Feng, and X. Wu (1997). *Modern Physics Letters A*, **12**, 1327.
Y. P. Yang, *et al.* (1998). *Modern Physics Letters A*, **13**, 879.